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Nuclear Forces and Neutron Stars

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Earlier work on nuclear forces is applied to a study of cooled massive neutron stars. Nuclear forces inside these stars cannot be neglected and their influence on neutron star states is considered. One important property of nuclear forces is their ability to be repulsive, which results in the phenomenon of nuclear saturation. It is shown that this property can provide the balance of gravitational and nuclear forces in cooled massive neutron stars.

1. INTRODUCTION

The problem of neutron star equilibrium was studied in the classical article of Oppenheimer and Volkoff (1939), who showed that there is no stationary solution of the general relativity equations corresponding to a cold neutron star having a mass bigger than $\sim 0.7M$. But this study of the problem cannot be regarded as complete because nuclear forces in stars were not taken into consideration. However, the nuclear forces in massive neutron stars can be essential when the stars become extremely cold and therefore shrink considerably.

To fill the gap in the study of this problem let us turn to a theory of nuclear forces (Rabinowitch, 1994, 1997). According to this theory, the nuclear field can be described by a scalar potential φ which satisfies the following covariant equation in an arbitrary coordinate system x^n:

$$(-g)^{-1/2}\partial[(-g)^{1/2}\partial\varphi/\partial x_n]/\partial x^n + (m_\pi c/\hbar)^2\varphi = -4\pi(G/m_p)^2\,\rho(\varphi) \quad (1)$$

$$g = \det(g_{ik}), \qquad \rho(\varphi) = \rho_0\,\exp(\varphi/c^2) \quad (2)$$

where g_{ik} are the components of the metric tensor, $\rho(\varphi)$ is the density of the nuclear matter mass at rest in a local inertial coordinate system, $\rho_0 = \rho(0)$,

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m_π and m_p are the masses at rest of the neutral pion and proton, respectively, and G is the constant of strong interaction.

The energy-momentum tensors T_m^{ik} and T_f^{ik} of a dustlike and uncharged matter and of the nuclear field generated by it, respectively, and the total energy-momentum tensor T^{ik} are as follows (Rabinowitch, 1994):

$$T_m^{ik} = c^2\,\rho_0 \exp(\varphi/c^2)\; dx^i/ds\; dx^k/ds$$

$$T_f^{ik} = (m_p^2/4\pi G^2)[(g^{in}g^{id} - \tfrac{1}{2}g^{ik}g^{nl})\;\partial\varphi/\partial x^n\;\partial\varphi/\partial x^l \tag{3}$$

$$^{\mathrm{f}}= +\; g^{ik}(m_\pi c/\hbar)^2\varphi^2/2]$$

$$T^{ik} = T_m^{ik} + T_f^{ik}, \qquad \nabla_i\, T^{ik} = 0, \qquad ds^2 = g_{ik}\, dx^i\, dx^k \tag{4}$$

For neutron stars, in which there are pressures and heat flows, we have to generalize expression (3) for the energy-momentum tensor of matter T_m^{ik}. This generalization will be given in the next section.

2. GRAVITATIONAL AND NUCLEAR FIELD AND THERMODYNAMIC EQUATIONS FOR NEUTRON STARS

Let us find the energy-momentum tensor T_m^{ik} of a nuclear matter with pressures and heat flows. For the matter in a comoving local inertial coordinate system we have the following expressions for the components T_m^{ik}:

$$T_m^{\infty} = c^2\rho_0 e^{\varphi/c^2} + u, \quad T_m^{0\alpha} = q_0^\alpha/c, \quad T_m^{\alpha\beta} = p\delta_{\alpha\beta}, \quad \alpha, \beta = 1, 2, 3$$

$$dx^\alpha/ds = 0, \quad ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \tag{5}$$

where we have used formula (2), and $\delta_{\alpha\beta}$ is the Kroneker symbol, p is the pressure in the matter, q_0^α is the three-dimensional vector of heat flow, and u is the density of the internal energy of the matter associated with its thermal motion.

It is easily seen that the tensor T_m^{ik} that satisfies (5) has the following form in an arbitrary coordinate system x^n:

$$T_m^{ik} = (c^2\rho_0 e^{\varphi/c^2} + u + p)\; dx^i/ds\; dx^k/ds - pg^{ik} + (q^i dx^k/ds$$

$$+\; q^k dx^i/ds)/c \tag{6}$$

where q^i is the four-dimensional vector of heat flow, which has the following components in a comoving local inertial coordiante system:

$$q^0 = 0, \qquad q^a = q_0^a, \qquad \alpha = 1, 2, 3, \qquad dx^a/ds = 0$$

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \tag{7}$$

Let us calculate the covariant derivative $\nabla_i\, T_m^{ik}$, taking into account the differential equation

$$\nabla_i(\rho_0\, dx^i/ds) = 0 \tag{8}$$

which expresses the conservation of a mass at rest m_0 having the density ρ_0 (Rabinowitch, 1994).

From (6) and (8) we obtain

$$\begin{aligned}\nabla_i T_m^{ik} = {} & \{e^{\varphi/c^2}\, \partial\varphi/\partial x^i + \partial[(u+p)/\rho_0]/\partial x^i\}\rho_0\, dx^i/ds\, dx^k/ds \\ & + (c^2\rho_0 e^{\varphi/c^2} + u + p)\nabla_i(dx^k/ds)\, dx^i/ds - \partial p/\partial x_k \\ & + [\nabla_i(q^k/\rho_0)\rho_0\, dx^i/ds \\ & + \nabla_i q^i\, dx^k/ds + q^i\nabla_i(dx^k/ds)]/c\end{aligned} \tag{9}$$

Since (Landau and Lifshitz, 1971)

$$\nabla_i(dx^k/ds)\, dx^i/ds = d^2x^k/ds^2 + \Gamma^k_{mn}\, dx^m/ds\, dx^n/ds \tag{10}$$

where Γ^k_{mn} are the Christoffel symbols, from (9) we get

$$\begin{aligned}\nabla_i T_m^{ik} = {} & (c^2\rho_0 e^{\varphi/c^2} + u + p)(d^2x^k/ds^2 + \Gamma^k_{mn}\, dx^m/ds\, dx^n/ds) \\ & + \rho_0 e^{\varphi/c^2} d\varphi/ds\, dx^k/ds + d[(u+p)/\rho_0]/ds\, \rho_0\, dx^k/ds - \partial p/\partial x_k \\ & + [\nabla_i(q^k/\rho_0)\rho_0\, dx^i/ds + \nabla_i q^i\, dx^k/ds + q^i\nabla_i(dx^k/ds)]/c\end{aligned} \tag{11}$$

As for the energy-momentum tensor T_f^{ik} of the nuclear field, its covariant derivative has the form

$$\nabla_i T_f^{ik} = -\rho_0 \exp(\varphi/c^2)\, \partial\varphi/\partial x_k \tag{12}$$

Formula (12) easily follows from (1) – (3) in local inertial coordinate systems (Rabinowitch, 1994). Since both sides of this formula are tensors, it is also true in arbitrary coordinate systems.

From (4), (11), and (12) we obtain the following dynamic equation:

$$\begin{aligned}\nabla_i T^{ik} = {} & (c^2\rho_0 e^{\varphi/c^2} + u + p)(d^2x^k/ds^2 + \Gamma^k_{mn}\, dx^m/ds\, dx^n/ds) \\ & + \rho_0 e^{\varphi/c^2}(d\varphi/ds\, dx^k/ds - \partial\varphi/\partial x_k) + [d(u+p)/ds - (1/\rho_0) \\ & \times (u+p)\, d\rho_0/ds]\, dx^k/ds - \partial p/\partial x_k + [\nabla_i(q^k/\rho_0)\rho_0\, dx^i/ds \\ & + \nabla_i q^i\, dx^k/ds + q^i\nabla_i(dx^k/ds)]/c = 0\end{aligned} \tag{13}$$

Let us use the equalities

$$(d^2x^k/ds^2 + \Gamma^k_{mn} dx^m/ds\, dx^n/ds)\, dx_k/ds = 0 \qquad \nabla_i(dx^k/ds)\, dx_k/ds = 0 \tag{14}$$

These equalities are obvious in local inertial coordinate systems because of the formulas

$$\frac{dx_k}{ds}\frac{d^2x^k}{ds^2} = \frac{1}{2}\frac{d}{ds}\left(\frac{dx^k}{ds}\frac{dx_k}{ds}\right) = 0$$

$$\frac{dx_k}{ds}\frac{\partial}{\partial x^i}\left(\frac{dx^k}{ds}\right) = \frac{1}{2}\frac{\partial}{\partial x^i}\left(\frac{dx^k}{ds}\frac{dx_k}{ds}\right) = 0$$

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \tag{15}$$

Since both sides of equalities (14) are tensors (Landau and Lifshitz, 1971), they are also true in arbitrary coordinate systems.

Multiply equation (13) by dx_k/ds. Then from (13) and (14) we obtain

$$du/ds - (1/\rho_0)(u + p)\, d\rho_0/ds$$
$$+ [\nabla_i q^i + \rho_0 \nabla_i (q^k/\rho_0)\, dx^i/ds\, dx_k/ds]/c = 0 \tag{16}$$

In order to understand the physical essence of equality (16), let us choose a comoving local inertial coordinate system. Then in this coordinate system from (16) and (7) we get

$$d(u + 2q^0/c)/dt - (1/\rho_0)(u + p)\, d\rho_0/dt + \partial q^\alpha/\partial x^\alpha = 0$$
$$dt = ds/c, \qquad dx^\alpha/ds = 0, \qquad \alpha = 1, 2, 3 \tag{17}$$

Let V_S be a small three-dimensional volume in the comoving local inertial coordinate system. Then

$$\rho_0 V_S = dm_0 = \text{const} \tag{18}$$

where dm_0 is a small invariable mass at rest.

From (18) we find

$$d(\rho_0 V_S)/dt = 0, \qquad d\rho_0 = -(\rho_0/V_S)\, dV_S \tag{19}$$

and from (17) and (19) we get

$$V_S\, d(u + 2q^0/c) + (u + p)\, dV_S + V_S\, dt\, \partial q^\alpha/\partial x^\alpha = 0, \quad \alpha = 1, 2, 3 \tag{20}$$

Taking into account (7), we can represent (20) in the form

$$d[(u + 2q^0/c)V_S] = -p\, dV_S - V_S dt\, \partial q^\alpha/\partial x^\alpha \tag{21}$$

Let us put

$$U = (u + 2q^0/c)V_S, \qquad \delta A = -p\, dV_S$$

$$\delta Q = -V_S\, dt\, \partial q^\alpha/\partial x^\alpha = -dt \int_{V_S} \partial q^\alpha/\partial x^\alpha\, dV_S = -dt \int_{S\delta} q_n\, dS_S \tag{22}$$

Here, as follows from (6), U is the internal energy of the small volume V_S,

δA is the work done by the pressure p, δQ is the heat entering the volume V_S through its surface S_S for the small time dt, and q_n is the projection of the vector q^α onto the outer normal of the surface S_S.

Hence from formulas (21) and (22) we get the first law of thermodynamics

$$dU = \delta A + \delta Q \tag{23}$$

This thermodynamic law just presents the physical essence of equality (16).

It is worth noting that the thermodynamic law (23) is a consequence of the differential laws of the conservation of energy, momentum, and mass at rest, (4) and (8), and of the nuclear field equation (1).

Let us turn to the gravitational field equations

$$R^i_k - \tfrac{1}{2} R g^i_k = \kappa T^i_k \tag{24}$$

and consider the stationary and spherically symmetric state of a neutron star.

Then the interval ds can be represented in the form

$$ds^2 = e^{\nu}(dx^0)^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) - e^{\lambda} dr^2 \tag{25}$$

where $\nu = \nu(r)$, $\lambda = \lambda(r)$, and r, ϕ, θ are spherical coordinates. In this case equations (24) are reduced to the following three equations (Landau and Lifshitz, 1971):

$$\kappa T^0_0 = e^{-\lambda}(\lambda'/r - 1/r^2) + 1/r^2, \qquad \kappa T^1_1 = -e^{-\lambda}(\nu'/r + 1/r^2) + 1/r^2$$

$$\nabla_i T^i_1 = 0 = (T^1_1)' + (2/r)(T^1_1 - T^2_2) + (\nu'/2)(T^1_1 - T^0_0) \tag{26}$$

where $x^1 = r$, $x^2 = \phi$, $x^3 = \theta$. For the other components T^i_k we have

$$T^3_3 = T^2_2, \qquad T^i_k = 0, \qquad i \neq k \tag{27}$$

In the stationary case under consideration, taking into account (27), $T^0_\alpha = 0$, $\alpha \neq 0$, we have

$$q^\alpha = 0, \qquad dx^\alpha/ds = 0, \qquad \alpha = 1, 2, 3 \tag{28}$$

Therefore, from (3), (6), (25), and (28) we find

$$T^0_0 = c^2\rho_0 e^{\varphi/c^2} + u + 2e^{\nu/2}q^0/c + (m_p^2/8\pi G^2)[e^{-\lambda}\varphi'^2 + (m_\pi c\varphi/\hbar)^2]$$

$$T^1_1 = -p - (m_p^2/8\pi G^2)[e^{-\lambda}\varphi'^2 - (m_\pi c\varphi/\hbar)^2]$$

$$T^2_2 = T^3_3 = -p + (m_p^2/8\pi G^2)[e^{-\lambda}\varphi'^2 + (m_\pi c\varphi/\hbar)^2)] \tag{29}$$

In the considered case, equation (1) for the nuclear potential φ acquires the form

$$\varphi'' + 2\varphi'[1/r + (\nu - \lambda)'/4] = e^{\lambda}[(m_\pi c/\hbar)^2\varphi + 4\pi(G/m_p)^2 \rho_0 e^{\varphi/c^2}] \tag{30}$$

3. COOLED NEUTRON STARS WITH ULTRALOW TEMPERATURES

Consider an arbitrary movement of a cooled neutron star with an absolute temperature $T \to 0$. In this case nuclear forces are essential since stars have to shrink considerably when their temperature becomes extremely low.

For this state of the star we have

$$\lim_{T\to 0} u = 0, \qquad \lim_{T\to 0} q^i = 0, \qquad i = 0, 1, 2, 3 \tag{31}$$

Equalities (31) follow from the fact that the matter internal energy and heat flows, which are described by the density u and vector q^i, respectively, and assotiated with thermal motions, are absent when the absolute temperature T is zero.

From (16) and (31) we easily find that also

$$\lim_{T\to 0} p = 0 \tag{32}$$

Let us return to the stationary spherically symmetric case. Then, when the absolute temperature is very close to zero, from (29), (31), (32), and (13) we get

$$\begin{aligned} T_0^0 &= c^2\rho_0 e^{\varphi/c^2} + (m_p^2/8\pi G^2)[e^{-\lambda}\varphi'^2 + (m_\pi c\varphi/\hbar)^2] \\ T_1^1 &= (m_p^2/8\pi G^2)[(m_\pi c\varphi/\hbar)^2 - e^{-\lambda}\varphi'^2] \\ T_2^2 &= T_3^3 = (m_p^2/8\pi G^2)\,[e^{-\lambda}\varphi'^2 + (m_\pi c\varphi/\hbar)^2] \end{aligned} \tag{33}$$

$$\nabla_i T_1^i = -c^2\rho_0 e^{\varphi/c^2}(\nu'/2 + \varphi'/c^2) = 0, \qquad 0 \le r \le r_0 \tag{34}$$

where $dx^\alpha/ds = 0$, $\alpha = 1, 2, 3$; $x^1 = r$, $x^2 = \phi$, $x^3 = \theta$.

Let r_0 be the radius of a cooled neutron star. Then from (34) we find

$$\varphi/c^2 = (\nu_0 - \nu)/2, \qquad \nu_0 = \text{const}, \qquad 0 \le r \le r_0 \tag{35}$$

Equations (2), (30), and (35) give the following formula for the mass density $\rho(\varphi)$.

$$\rho = \rho_0 e^{\varphi/c^2} = \rho_0 e^{(\nu_0 - \nu)/2} = -(m_p^2 c^2/8\pi G^2)\{(m_\pi c/\hbar)^2(\nu_0 - \nu) + e^{-\lambda}[\nu'' + \nu'(2/r + (\nu - \lambda')/2)]\}, \qquad 0 \le r \le r_0 \tag{36}$$

From (26), (33), (35), and (36) we get

$$1 - e^{-\lambda}(1 - \lambda' r) = -(\kappa m_p^2 c^4 r^2/8\pi G^2)[(m_\pi c/2\hbar)^2(\nu_0 - \nu)(4 + \nu - \nu_0)$$
$$+ e^{-\lambda}(\nu'' + 2\nu'/r - \nu'\lambda'/2 + \nu'^2/4)]$$
$$1 - e^{-\lambda}(1 + \nu' r) = -(\kappa m_p^2 c^4 r^2/32\pi G^2) \tag{37}$$
$$\times [e^{-\lambda}\nu'^2 - (m_\pi c/\hbar)^2(\nu_0 - \nu)^2], \qquad 0 \leq r \leq r_0$$

Consider $\lambda(r)$ and $\nu(r)$ at the point $r = 0$. As follows from (37), we have

$$\lambda(0) = 0 \tag{38}$$

By performing the differentiation at the point $r = 0$ of the two equations (37) and using (38), we get

$$2\lambda'(0) + (\kappa m_p^2 c^4/4\pi G^2)\nu'(0) = 0, \qquad \lambda'(0) - \nu'(0) = 0 \tag{39}$$

Hence from (38) and (39) we have

$$\lambda(0) = \lambda'(0) = \nu'(0) = 0 \tag{40}$$

Let us introduce the following variable x and functions $f(x)$ and $g(x)$, taking into account (40):

$$x = (r/r_0)^2, \; e^{-\lambda} = 1 - xf(x), \qquad \nu - \nu_0 = g(x), \qquad 0 \leq x \leq 1 \tag{41}$$

and put

$$\alpha = \kappa m_p^2 c^4/8\pi G^2, \qquad \beta = m_\pi c r_0/2\hbar \tag{42}$$

Then equations (37) can be represented in the form

$$2xf' + 3f = \alpha[4\beta^2 g - 2(1 - xf)(2xg'' + 3g' + xg'^2)$$
$$+ 2xg'(xf' + f) + (1 - xf)xg'^2 + \beta^2 g^2]$$
$$2(1 - xf)g' - f = \alpha\,[(1 - xf)xg'^2 - \beta^2 g^2] \tag{43}$$
$$f = f(x), \qquad g = g(x), \qquad 0 \leq x \leq 1$$

Since

$$\beta = 0.5 r_0/r_\pi, \qquad r_\pi = \hbar/m_\pi c \tag{44}$$

where r_0 is the star radius and r_π is the Compton length of the pion, the value β is very large:

$$\beta >> 1 \tag{45}$$

As β is enormous, equations (43) practically coincide with the equations

$$2xf' + 3f = \alpha\beta^2 g(4 + g), \qquad f - 2(1 - xf)g' = \alpha\beta^2 g^2 \tag{46}$$

It follows from (41) and (46) that we have to seek functions $f(x)$ and $g(x)$ bounded at $x = 0$ in order to satisfy condition (40). For such functions we derive the following condition from the first equation in (46) by putting $x = 0$:

$$f(0) = \alpha\beta^2 g(0)[4 + g(0)]/3 \tag{47}$$

Consider now equations (26), (30), and (33) when $r \geq r_0$. Outside a neutron star having the radius r_0 the density ρ_0 can be represented in the form (Rabinowitch, 1997)

$$\rho_0 = 2\sigma\delta(\overline{r - r_0}), \qquad r \geq r_0 \tag{48}$$

Here $\delta(x)$ denotes the even delta function, $\overline{r - r_0}$ is the distance between a point (r, ϕ, θ) and the sphere $r = r_0$ in a comoving local inertial coordinate system chosen near the sphere, and $\sigma = \text{const}$.

Formula (48) expresses the density of virtual pions created in the physical vacuum at the surface $r = r_0$ because of the influence on it of the surface. The value of the constant σ is given by the formula (Rabinowitch, 1997)

$$\sigma = 0.049/s, \qquad s = 4\pi\hbar G^2/m_p^2 m_\pi c^3, \qquad G^2/\hbar c = 0.080 \tag{49}$$

When $d\phi = d\theta = 0$, from (25) we have

$$ds^2 = e^{\nu}(dx^0)^2 - e^{\lambda} dr^2 = (d\bar{x}^0)^2 - d\bar{r}^2 \tag{50}$$

where $\bar{x}^0$, $\bar{r}$ are the corresponding coordinates in a comoving local inertial coordinate system.

As is well known, for the comoving coordinate system $\bar{x}^0$, $\bar{r}$ from (50) we have

$$d\bar{x}^0 = e^{\nu/2} dx^0, \qquad d\bar{r} = e^{\lambda/2} dr \tag{51}$$

From (48) and (51) we get

$$\rho_0 = 2\sigma\delta[e^{\lambda/2}(r - r_0)] = 2\sigma e^{-\lambda/2}\delta(r - r_0), \qquad r \geq r_0 \tag{52}$$

Formulas (30) and (52) give the following equation for the nuclear field potential φ outside the star:

$$\varphi'' + 2a\varphi'/r - b^2\varphi = w, \qquad \varphi = \varphi(r), \qquad r \geq r_0 \tag{53}$$

where

$$a = 1 + (\nu - \lambda)'r/4, \qquad b = e^{\lambda/2} m_\pi c/\hbar$$

$$w = 8\pi(G/m_p)^2\sigma e^{\lambda/2 + \varphi/c^2}\,\delta(r - r_0), \qquad \int_0^\infty \delta(x)\,dx = 1/2 \tag{54}$$

Equation (53) can be represented in the form

$$\varphi''(z) + 2a(z)\varphi'(z)/z - 4\beta^2 e^{\lambda(z)}\varphi(z) = r_0^2 w(z), \quad z = r/r_0, \quad z \geq 1 \tag{55}$$

where β is defined by formula (44), $\beta >> 1$, and

$$a(z) = 1 + [\nu'(z) - \lambda'(z)]z/4 \tag{56}$$

In order to solve equation (55), let us introduce two functions $p(z)$ and $q(z)$ which satisfy the equations

$$p + q = 2a(z)/z, \qquad p' + pq = -4\beta^2 e^{\lambda(z)} \tag{57}$$

Consider the function

$$y(z) = \varphi'(z) + p(z)\varphi(z) \tag{58}$$

Then from (57) and (58) we find

$$y' + qy = \varphi'' + (p + q)\varphi' + (p' + pq)\varphi = \varphi'' + 2a\varphi'/z - 4\beta^2 e^{\lambda}\varphi \tag{59}$$

and from (55) we get

$$y' + qy = r_0^2 w, \qquad z \geq 1 \tag{60}$$

Hence the differential equation (55) of the second order is equivalent to equations (57), (58), and (60) of the first order.

From (57) we have

$$(p_0' + 2ap_0/z)/\beta - p_0^2 = -4e^{\lambda}, \qquad p_0 = p/\beta \tag{61}$$

Since the value β defined by (44) is enormous, we find the following approximate solution of equation (61), which practically coincides with the exact one:

$$p_0^2 \approx 4e^{\lambda}, \qquad p_0 = p/\beta, \qquad \beta >> 1 \tag{62}$$

As follows from (57) and (62), we can put

$$p = 2\beta e^{\lambda/2}, \qquad q = 2(a/z - \beta e^{\lambda/2}) \approx -2\beta e^{\lambda/2} \tag{63}$$

From (58), (60), (63), and (54) we easily find the following solutions $\varphi(z)$ and $y(z)$ vanishing at infinity:

$$\begin{aligned} \varphi(z) &= \exp\left(-\int_1^z p\,dz\right)\left[\int_\infty^z y \exp\left(\int_1^z p\,dz\right)dz + D\right], \quad D = \text{const} \\ y(z) &= r_0^2 \exp\left(-\int_1^z q\,dz\right)\int_\infty^z w \exp\left(\int_1^z q\,dz\right)dz, \; \varphi(\infty) = y(\infty) = 0, \; z \geq 1 \end{aligned} \tag{64}$$

Formulas (58) and (64) give

$$\varphi'(1) + p(1)\varphi(1) = y(1) = -r_0^2 \int_1^\infty w \exp\left(\int_1^z q\, dz\right) dz,$$

$$\varphi = \varphi(z), \quad z = r/r_0 \tag{65}$$

From (54), (63), and (65) we obtain

$$\varphi'(1) + 2\beta e^{\lambda(1)/2}\varphi(1) = -4\pi r_0 \sigma (G/m_p)^2\, e^{\lambda(1)/2 + \varphi(1)/c^2},$$

$$\varphi = \varphi(z), \quad \lambda = \lambda(z) \tag{66}$$

Using again the variable $x = (r/r_0)^2 = z^2$, from (35), (41), and (66) we find the following condition at the point $x = 1$ ($r = r_0$):

$$g'(1) + \beta g(1)(1 - f(1))^{-1/2} = 4\pi r_0 \sigma (G/m_p c)^2 (1 - f(1))^{-1/2} e^{-g(1)/2}$$

$$g = g(x), \qquad f = f(x), \qquad x = (r/r_0)^2 \tag{67}$$

Recalling that β is very large, from (67) and (44) we get

$$g(1)e^{g(1)/2} \approx 4\pi r_0 \sigma (G/m_p c)^2/\beta = 8\pi\sigma\hbar G^2/m_{\mathrm{p}}^2 m_\pi c^3, \qquad \beta >> 1 \tag{68}$$

Formulas (49) and (68) give

$$g(1)e^{g(1)/2} = 0.098 \tag{69}$$

This equation has the solution

$$g(1) = 0.0935 \tag{70}$$

Consider now the functions $\lambda(r)$ and $\nu(r)$ when $r \geq r_0$. From (26) we obtain

$$e^{-\lambda} = 1 - r_g/r - (\kappa/r)\int_\infty^r r^2 T_0^0\, dr, \quad \nu = \int_\infty^r [(e^\lambda - 1)/r - \kappa r e^\lambda T_1^1]\, dr \tag{71}$$

Here

$$\kappa = 8\pi\gamma/c^4, \qquad r_g = 2\gamma M/c^2 \tag{72}$$

where γ is the gravitational constant, and r_g and M are the gravitational radius and mass of the neutron star, respectively.

When $r > r_0$, from (54), (63), and (64) we find

$$w(z) = 0, \qquad y(z) = 0, \qquad \varphi(z) = D \exp\left(-2\beta \int_1^z e^{\lambda/2}\, dz\right),$$

$$z = r/r_0, \qquad z > 1 \tag{73}$$

From (44) and (73) we get

$$\varphi'(r) = -(m_\pi c/\hbar)e^{\lambda/2}\varphi(r), \qquad r > r_0 \tag{74}$$

and from (33), (42), (52), and (74) we derive

$$T_0^0 = 2c^2\sigma e^{\varphi/c^2-\lambda/2}\,\delta(r - r_0) + 8\varphi^2/\varepsilon\kappa r_0^2 c^4, \quad T_1^1 = 0, \qquad r > r_0 \tag{75}$$

where

$$\varepsilon = 1/\alpha\beta^2 \tag{76}$$

From (42), (72), and (76) we get

$$\varepsilon^{-1} = \gamma(m_p m_\pi c r_0/2\hbar G)^2 = (r_0/r_g)^2(\gamma^{3/2} m_p m_\pi M/c\hbar G)^2 \tag{77}$$

Formula (77) can be represented in the form

$$\varepsilon^{-1/2} = \chi(M/M_\odot)(r_0/r_g), \qquad \chi = \gamma^{3/2} m_p m_\pi M_\odot/c\hbar G) \tag{78}$$

where γ is the gravitational constant, $M_\odot$ is the mass of the Sun, and χ is a dimensionless constant.

Using (49) for the constant G and the well-known values of the constants γ, m_p, m_π, c, $\hbar$, $M_\odot$, we can determine the constant χ. Its value is as follows

$$\chi = 0.2747 \tag{79}$$

Hence from (78) and (79) we get

$$\varepsilon^{-1/2} = 0.2747(M/M_\odot)(r_0/r_g) \tag{80}$$

When $r \geq r_0$ formulas (71) and (75) give

$$e^{-\lambda} = 1 - r_g/r + \kappa\, c^2 r_0 \sigma\, e^{\varphi/c^2 - \lambda/2}\, N(r - r_0) + (8/\varepsilon c^4 r_0^2 r) \int_r^\infty r^2\varphi^2\, dr, \qquad r \geq r_0 \tag{81}$$

where

$$N(0) = 1 \qquad N(r - r_0) = 0, \qquad r > r_0 \tag{82}$$

Since β is enormous, from (73) we get

$$\varphi(r) \approx 0, \qquad r > r_0, \qquad \beta >> 1 \tag{83}$$

Therefore, from (81) and (83) we find

$$e^{-\lambda} = 1 - r_g/r + \kappa c^2 r_0 \sigma e^{\varphi/c^2-\lambda/2}\, N(r - r_0), \qquad r \geq r_0 \tag{84}$$

From (42), (49), and (76) we have

$$\kappa c^2 r_0 \sigma = 0.049 m_p^2 m_\pi c^5 r_0 \kappa/4\pi\hbar G^2 = 0.196\alpha\beta = 0.196/\beta\varepsilon \tag{85}$$

where ε is determined by (80).

Taking again into account that β is enormous, from (84) and (85) we find

$$e^{-\lambda} \approx 1 - r_g/r, \qquad r \geq r_0, \qquad \beta >> 1 \tag{86}$$

From (71), (75), and (86) we obtain

$$\nu = -\int_r^\infty (e^\lambda - 1)/r\, dr \approx \ln(1 - r_g/r), \qquad r \geq r_0 \tag{87}$$

Hence from (25), (86), and (87) we get the Schwarzschild interval ds outside the neutron star.

When $r = r_0$, formulas (41), (70), (86), and (87) give

$$r_g/r_0 = f(1), \qquad \nu_0 = \ln(1 - r_g/r_0) - 0.0935 \tag{88}$$

Let us put

$$h(x) = \varepsilon f(x) \tag{89}$$

For the functions $h(x)$ and $g(x)$, from (46), (47), (70), (76), and (89) we get the following equations:

$$2xh'(x) + 3h(x) = g(x)(4 + g(x))$$

$$2(\varepsilon - xh(x))g'(x) = h(x) - g^2(x), \qquad 0 \leq x \leq 1 \tag{90}$$

$$g(0) = g_0, \qquad h(0) = g_0(4 + g_0)/3, \qquad g(1) = 0.0935 \tag{91}$$

where g_0 is some constant dependent on ε.

In order to determine g_0, we have to use the third condition in (91) for $g(1)$. The obtained equations (90) and (91) can be numerically solved for different values of the parameter ε.

Thus the problem under consideration is reduced to equations (90) and (91). After obtaining computer numerical solutions of these equations we can find the functions $\lambda(r)$, $\nu(r)$, $\varphi(r)$, and $\rho(r)$ inside the neutron star by formulas (35), (36), (41), (88), and (89).

The values of r_0/r_g and $M/M_\odot$ can be determined by formulas (80), (88), and (89) as follows:

$$r_0/r_g = \varepsilon/h(1), \qquad M/M_\odot = 3.640\, h(1)/\varepsilon^{3/2} \tag{92}$$

Here r_g and M are the gravitational radius and mass of the neutron star, respectively.

We have performed computer numerical integration of equations (90) and (91) for different values of the parameter ε. For each ε the parameter g_0 was also varied and was chosen so that the condition $g(1) = 0.0935$ in (91) was satisfied.

Table I. Computer Constant g_0, Radius r_0, and Mass M of Cooled Neutron Stars for Different Values of the Parameter ε

ε	$g_0 \times 10^4$	$M/M_\odot$	r_0/r_g
10.000	877.984820	0.014	80.3936
4.000	799.811353	0.055	33.3656
2.000	687.477403	0.146	17.6730
1.000	515.679144	0.372	9.7918
0.500	306.338906	0.891	5.7804
0.200	86.293776	2.540	3.2049
0.100	17.358031	5.231	2.2003
0.070	5.661120	7.433	1.8509
0.050	1.562563	10.219	1.5929
0.040	0.576383	12.527	1.4529
0.030	0.131147	16.124	1.3034
0.025	0.045177	18.796	1.2248
0.020	0.010540	22.492	1.1444
0.017	0.003255	25.464	1.0963
0.015	0.001225	27.894	1.0655
0.013	0.000369	30.790	1.0368
0.012	0.000181	32.448	1.0241
0.011	0.000081	34.258	1.0131
10.010	0.000032	36.233	1.0046

The results of the computer calculations of the solutions of equations (90) and (91) and then of the values r_0/r_g and $M/M_\odot$ are given in Table I.

As follows from Table I, equations (90) and (91) have solutions for large values of the parameter $M/M_\odot$. This means that nuclear forces can balance gravitational forces in cooled massive neutron stars.

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